LIGHT CURVE OF A SOURCE ORBITING AROUND
A BLACK HOLE: A FITTING-FORMULA

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ABSTRACT

A simple, analytical fitting-formula for a photometric light curve of a source of light orbiting around a black hole is presented. The formula is applicable for sources on a circular orbit with radius smaller than 45 gravitational radii from the black hole. This range of radii requires gravitational focusation of light rays and the Doppler effect to be taken into account with care. The fitting-formula is therefore useful for modelling the X-ray variability of inner regions in active galactic nuclei.

Subject headings: galaxies: active — accretion, accretion disks — X-rays: galaxies — black hole physics

1. INTRODUCTION

In their seminal paper, Cunningham & Bardeen (1973) studied photometric light curves of a point-like source of light orbiting in the equatorial plane of a rotating (Kerr) black hole. These authors adopted approximation of geometrical optics and presented a detailed discussion of periodic variations of the observed frequency-integrated flux. Origin of the variability is twofold: (i) energy of individual photons is affected by the Doppler effect (the effect of special relativity), and (ii) photon trajectories are influenced by the gravitational field of the central black hole (purely general-relativistic effect). Motivation to study relativistic effects on light from a source near a black hole comes from astronomy, however, no direct observational comparisons were possible in the early 1970s when the formalism was first established.

During the last decade the interest of astronomers in modelling detailed features in the light curve of an orbiting source has revived. Satellite monitoring of active galactic nuclei (AGNs) focused our attention to X-ray variability (Duschl, Wagner & Camenzind 1991; Miller & Wiita 1991) as a means of investigating the central object, presumably a massive black hole surrounded by an accretion disk (Rees 1984). Variability time-scales indicate that X-rays originate in the innermost region of a few gravitational radii: $R \lesssim 25\,R_g$ (e.g., Heeschen et al. 1987). Power spectrum of AGNs has a particularly complex, featureless behavior at frequency $\omega \gtrsim 10^{-5}\,\mathrm{Hz}$. The fluctuating signal can be represented, in the frequency domain, by a power-law: $F_\omega \propto \omega^{-\alpha}$ with $1 \lesssim \alpha \lesssim 2$ (Lawrence et al. 1987). No strictly periodic AGNs have been discovered but the power spectrum of some objects (Papadakis & Lawrence 1995) contains excessive power at frequency $\omega \gtrsim 10^{-2}\,\mathrm{Hz}$, indicating quasi-periodic oscillations (QPOs) analogous to those associated with low-mass X-ray binaries (Lewin & Tanaka 1995).

In spite of the evident progress on the observational side, theoretical understanding of the short-term variability of AGNs remains rather insufficient. Several physical mechanisms have been proposed, referring to instabilities in accretion disks and associated jets. Possible existence and properties of vortices on accretion disks have been discussed by Abramowicz et al. (1992) and Adams & Watkins (1995). It is speculated that these vortices survive over several (perhaps many) local orbital periods. Radiation from separate vortices is modulated by their orbital motion around the black hole and a large number of individual contributions results in observed fluctuations. In a more phenomenological approach, one refers to spots (localized regions of enhanced or diminished emissivity) residing on the surface of the disk (Abramowicz et al. 1991; Wiita et al. 1991; Zhang & Bao 1991). It is not crucial for the form of the power spectrum whether the spots are identified with vortices or whether they are of different physical nature. Our present contribution deals with the bright-spot model

but we would like to emphasize that other scenarios for the short-term variability of AGNs have been proposed. Krolik et al. (1991) explored ultraviolet continuum variability and pointed out that independent fluctuations originating at different radii in the source can produce power-law spectra. Mineshige, Ouchi & Nishimori (1994) employed the theory of self-organized critical states that develop and persist in the disk material, also leading to the observed power spectra. Kanetake, Takeuti & Fukue (1995) proposed that vertical oscillations of accretion disks are the source of QPOs and studied characteristic periods of these oscillations. Ipser (1994) and Perez et al. (1996) associated QPO phenomena with frame-dragging effects which determine behavior of relativistic disks, and in particular with low-frequency modes in accretion disks near a Kerr black hole. Lamb et al. (1985), in the context of Galactic QPO sources, explored interaction of a clumpy disk with magnetosphere around a central object. At present, it is probably fair to say that the relation between various possible instabilities and the resulting variable signal of AGNs is, at the best, only vaguely understood. It is thus impossible to discriminate between intrinsically different models, even when they are testable in principle and the necessary information is contained in the observed signal. In other words, it appears encouraging that some features of the power spectra can be explained by the bright-spot model but, in reality, this fact does not mean very much when the emission properties of individual spots are largely unknown. One can say (with Antoine de Saint-Exupéry) that "Truth is not that which is demonstrable, but that which is ineluctable."

Until the theory of (nonaxisymmetric) instabilities in accretion disks yields specific results about radiation outgoing from the disk surface, phenomenological models must take all possible values of their parameter space into account. This has not yet been possible to carry out with the bright-spot model which has a number of degrees of freedom: distribution of the spots across the disk, intrinsic emissivity properties of individual spots, inclination of the observer, etc. Several numerical codes evaluating the light-curve profiles and corresponding power spectra have been developed (Asaoka 1989; Bao 1992; Karas, Vokrouhlick & Polnarev 1992; Zakharov 1994) but models with a large number of different spots still remain too expensive computationally (some steps in the original derivation given by Cunningham & Bardeen [1973] need to be evaluated numerically). The authors had to impose additional assumptions on the model without a sound physical reason (a "canonical" number of 100 spots is just an example of such restriction). The main aim of the present contribution is to approximate the light curve of an individual orbiting source of light by a practical fitting-formula. It is required that the formula, while reflecting relativistic effects in the light curve, can be implemented in a fast code and substitute extensive ray tracing which is otherwise necessary. It will be specified later what can be considered as "practical" and what is a suitable approximation for our purpose.

The fitting-formula makes it possible to overcome unnatural restrictions which are routinely imposed on the bright-spot model, and to cover the parameter space of various models in a much more complete manner. The fitting-formula is given in the next section.

2. THE FITTING-FORMULA

2.1. Assumptions

It is assumed that the source of light is located on a circular orbit in the equatorial plane of a Kerr black hole (Misner, Thorne & Wheeler 1973). Prograde orbits of the source (co-rotating with the black hole) and direct-image trajectories of photons (not crossing equatorial plane) are considered. Observer's position is specified, in Boyer-Lindquist coordinates, by $R \to \infty$, $\theta = \theta_0$. (Azimuthal coordinate is arbitrary due to axial symmetry.) Observed radiation flux is determined by propagating photons along null geodesics from the source to the observer, in accordance with approximation of geometrical optics. It is straightforward to calculate the light curve of a point-like source. The light curve of a finite-size source can be obtained by integrating over the surface of the source. Calculation of the light curves of finite-size sources contributes significantly to the total computational cost of the bright-spot model and we wish to make this step much faster. Technical details of our code which calculates light-curve profiles were described by Karas et al. (1992).

Standard notation (e.g., Misner et al. 1973) will be adopted throughout this work and geometrized units ($c = G = 1$) will be used; M denotes the mass of the central black hole, a is its dimensionless angular-momentum parameter. All lengths and times are made dimensionless by expressing them in units of M. Gravitational radius of the Kerr black hole is expressed in terms of a: $R_g = 1 + (1 - a^2)^{1/2}$, $0 \leq a \leq 1$.

It turns out that the light-curve profile is very sensitive to observer inclination, θ_0, and radius of the orbit of the source, R_s. We will thus focus our attention to these two parameters. Rauch & Blandford (1994) studied light curves of point-like sources which show extremely high peaks when the source crosses a caustic. Shape of the caustic depends on angular momentum of the black hole and the light curve is thus sensitive also to the value of a. The case is different when a finite-size source which covers the whole caustic is considered (or, alternatively, temporal resolution of observation is lower than the caustic crossing-time); the caustic is then unresolved and the high-magnification spikes are smoothed down. We will consider only a situation when the source does not cross the caustic or when its size,

d, exceeds the cross-sectional size of the caustic, s. For $R_s \gg 1$, equation (6) of Rauch & Blandford (1994) gives an asymptotic formula: $s \approx 0.34\, R_s^{-1} a^2 \sin^2 \theta_0$. In our calculations, it was assumed that each spot radiates isotropically in its local comoving frame and that the local emissivity decreases exponentially with the distance from the center of the spot; we checked that the computed profiles depend only weakly on these assumptions when the size of the spot satisfies condition $s \ll d \ll R_s$.

It is only a variable component of the signal which is relevant for the source fluctuations. Zero-level of our light curves will thus be set at minimum of the observed signal. Variable component of the radiation flux (counts per second) will be given in arbitrary units. This arbitrary scaling of the profile contains sufficient information about the light curve when radiating spots are all located at a constant radius. Additional information about the absolute value of the maximum counting rate is necessary when a distribution of spots at different radii is considered. In this case one has to prescribe total flux from individual spots and their geometrical shape as a function of radius. Since we pay little attention to the model-dependent quantities, the absolute scaling of the counting rate is not discussed in the present contribution (except for an illustrative example mentioned in next section).

Analogously to the absolute value of the counting rate, also the phase of the light curve is arbitrary. We scale the phase to interval $0 \leq \varphi \leq 1$ and define the phase to be equal to 0.5 for the maximum magnification due to the focusing effect. This definition means that phase 0.5 corresponds to photons coming approximately (exactly when $a = 0$) from the opposite side of the orbit, across rotation axis to the observer. The phase φ increases linearly with time, as measured by a distant observer. Relation between φ and the orbital phase of the spot on the disk surface is complicated due to time-delay, but it has been taken into account automatically by integrating individual photon trajectories. Maximum Doppler enhancement of the observed radiation corresponds to phase ≈ 0.75. Dimensionless phase can be easily converted to time interval (as measured by a distant observer) when a rotation law of the spots is specified. For example, $\Delta t = R_s^{3/2} + a$ is the period of the Keplerian circular orbit (as measured by a distant observer) which is appropriate for a thin disk in the equatorial plane.

2.2. Numerical Method and Results

A series of the light-curve profiles corresponding to a spot orbiting in the range of distances $R_s \leq 45\, R_g$ were computed. A model function $F(\varphi, R_s, \theta_0; p_k)$ was specified, depending nonlinearly on unknown parameters p_k, $k = 1, \ldots, 8$. (F is the measured

frequency-integrated flux in arbitrary units.) The form of this function has been chosen on the basis of asymptotic behavior and our experience with extensive calculations of the profiles (Karas et al. 1992):

$$
\begin{aligned}
F &= \left(\frac{p_3 \cos \theta_0}{R_s} + p_7 \, (R_s - 1)^{2/5} \right) \left[1 + \sin \left[2\pi (\varphi + p_4 \, R_s - \tfrac{3}{5}) + \tfrac{\pi}{2} \right] \right]^z \cos^{-2/3} \theta_0 \\
&+ \left(p_1 + p_6 R_s^{1/3} \right) \exp \left[-p_2 \left| \varphi - \tfrac{1}{2} \right|^{9/5} \right] \cos^{-2} \theta_0,
\end{aligned}
\tag{1}
$$

where

$$
z = p_5 \cos^{1/2} \theta_0 + p_8 \cos^{3/2} \theta_0.
$$

Parameters p_k were determined by the Levenberg-Marquardt method (nonlinear least-square fitting; Press et al. 1994). The range of fitting was restricted to

$$
20° \le \theta_0 \le 80°, \qquad 3 \, R_g \le R_s \le 45 \, R_g,
\tag{2}
$$

which is where relativistic effects on the light curve are most profound. Table 1 gives the results of the fitting for two values of the black-hole angular momentum: $a = 0$ (a nonrotating, Schwarzschild black hole), and $a = 1$ (an extremely rotating Kerr black hole). The a-dependence is only weak and it reflects a shift of relative phases of the two peaks in the light curve—the Doppler and the focusation peak (both peaks are simultaneously visible only if $\theta_0 \gtrsim 70°$, however). The shift is caused by the frame-dragging effect which operates near rotating black holes. Figure 1 illustrates a typical form of the profiles normalized to the maximum flux. The curves have been plotted according to equation (1) with parameters taken from Table 1.

Until now, only quantities which depend weakly on local properties of individual spots were discussed. However, some applications are strongly model-dependent, for example when radial distribution of spots or eclipses of spots by a thick disk are taken into account (Bao & Stuchlík 1992; Karas & Bao 1992; Mangalam & Wiita 1993). Vortices located across the surface of an accretion disk (as proposed by Abramowicz et al. 1992) can act as spots in our model. Maximum flux from spots at different radii must then be specified. Figure 2 illustrates the maximum flux as a function of radius for our simple model of isotropically radiating spots with constant size d. Relative fluxes from spots orbiting at different radii can be obtained by scaling the normalized light curves from Fig. 1 by a corresponding value of the maximum flux. It is evident from Fig. 2 that, at small radii, gravitational redshift decreases the total flux when $\theta_0 \lesssim 50°$, while the Doppler and the lensing enhancement dominate for $\theta_0 \gtrsim 50°$. It is straightforward to obtain graphs analogous to Fig. 2 also for other models of the spot local emissivity.

3. CONCLUSION

Variable signal from a source orbiting around a black hole is approximated by the fitting-formula (1). This formula presents a simple *model* of the corresponding exact expression which appears too complex and inconvenient for numerous applications. Within the above-discussed range of validity of the formula, the normalized light-curve profile is nearly independent of the source shape. Approximations which have been adopted turn out to be adequate in many astrophysically relevant situations, e.g. in exploring direct images of finite-size spots. This subject has been explored by numerous authors but the fitting-formula offers much faster opportunity for studying light-curve profiles. It is suggested that the formula is very practical for modelling short-term featureless X-ray variability and quasi-periodic oscillations in active galactic nuclei within framework of the bright-spot model (work in progress).

TABLE 1

PARAMETERS OF THE FITTING-FORMULA

a	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
0	0.021696	190.7236	0.3476	-0.0018	3.5106	-3.6×10^{-5}	0.0124	-0.0231
1	0.024258	181.8421	0.0958	-0.0032	4.7862	-3.0×10^{-5}	0.0109	-0.1527

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FIGURE CAPTIONS (Figures available upon request from the author)

Figure 1—Illustration of the fitting-formula. The light-curve profile normalized to the maximum flux (the counting rate) is shown by solid lines for three cases: (a) $R_s = 3\,R_g$, $\theta_0 = 80°$; (b) $R_s = 44\,R_g$, $\theta_0 = 80°$; (c) $R_s = 44\,R_g$, $\theta_0 = 20°$. Shapes of the normalized curve depend on a only weakly; here $a = 1$.

Figure 2—Maximum flux (in arbitrary units) from a spot as a function of dimensionless radius R_s/R_g of the orbit; (a) $a = 0$, (b) $a = 1$. Observer inclination is indicated by three symbols in each of the graphs: "•"... $\theta_0 = 20°$, "+"... $\theta_0 = 50°$, "∗"... $\theta_0 = 80°$.